

## RADIAL SOLUTION OF THE MONGE-AMPÈRE EQUATION ON THE COMPLEX GRASSMANNIAN $G_{2,4}\mathbb{C}$

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**ABSTRACT.** In this paper we determine a radial solution of the Monge-Ampère equation on  $G_{2,4}\mathbb{C}$  using a change of symplectic variable which helps us to simplify the writing of the latter then put it in radial form which will result in an ordinary differential equation.

**Keywords:** Calabi-Yau structure, Monge-Ampère equation, Radial solution.

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### INTRODUCTION

In dimension  $n$ , a Monge-Ampère equation is a linear combination of the minors of the Hessian matrix [2]

$$Hess(f) = \left( \frac{\partial^2 f}{\partial q_j \partial q_k} \right)_{j,k=1 \dots n}.$$

Let  $\Omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \dots + dz_n \wedge d\bar{z}_n)$  be the canonical Kähler form on  $\mathbb{C}^n$  and  $\alpha = dz_1 \wedge \dots \wedge dz_n$  be the complex volume form. A special lagrangian submanifold is a real  $n$ -submanifold  $L$  which is lagrangian with respect to  $\Omega$  and which satisfies the special condition

$$Im(\alpha)|_L = 0.$$

These submanifolds, introduced by Harvey and Lawson in their famous article *Calibrated Geometries* [4] are minimal submanifolds of  $\mathbb{C}^n$ , and more generally of Calabi-Yau manifolds, transverse in some sense to complex submanifolds. They have been extensively studied after the construction proposed by Strominger, Yau and Zaslow [6] of mirror partners of Calabi-Yau manifolds based upon an hypothetic special lagrangian fibration.

Some examples have been given by many people. We can cite for example Harvey and Lawson [4], Joyce [5] and Bryant [3].

For every smooth function  $f$  on  $\mathbb{R}^n$ , the graph

$$L_f = \left\{ \left( q + i \frac{\partial f}{\partial q} \right), q \in \mathbb{R}^n \right\}$$

is a lagrangian submanifold of  $\mathbb{C}^n$ . The special lagrangian condition becomes then a differential equation on  $f$ , called the special lagrangian equation: Noting that the Hessian matrix is symmetric, this is due to the formula [2]:

$$Im(\det_{\mathbb{C}}(Id + iA)) = \sum_{k=0}^n (-1)^k \sigma_{2k+1}(A) = 0$$

where  $A$  is a symmetric matrix.

$$\sum_{k=0}^n (-1)^k \sigma_{2k+1}(\text{Hess}(f)) = 0$$

For  $n = 3$ : (see [1])

$$\Delta f - \text{hess}(f) = 0$$

A symplectic Monge-Ampère equation in dimension 3 associated with a non-degenerate Monge-Ampère structure  $(\Omega, \omega)$  can be brought back by a change of variables symplectic to one of the following equations

$$\begin{cases} \text{hess}(f) = 1 \\ \Delta f - \text{hess}(f) = 0 \\ \square f + \text{hess}(f) = 0 \end{cases}$$

is integrable if and only if

$$\begin{cases} d\left(\frac{\omega}{\sqrt[4]{|\lambda(\omega)|}}\right) = 0 \\ q \text{ is flat} \end{cases}$$

in dimension 3:

$$\Delta f - \text{hess}(f) = 0 \Leftrightarrow \omega = dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 - dp_1 \wedge dp_2 \wedge dp_3$$

and for the Hessian equation:

$$\text{hess}(f) = 1 \Leftrightarrow \omega = dp_1 \wedge dp_2 \wedge dp_3 - dq_1 \wedge dq_2 \wedge dq_3$$

The geometric structure on  $T^*\mathbb{R}^3$  associated with the Lagrangian special equation

$$\Delta f - \text{hess}(f) = 0$$

is the classic Calabi-Yau structure  $(q, I, \Omega, \alpha)$  with:

$$\begin{cases} q = -\sum_{j=1}^3 dx_j \cdot dx_j + dy_j \cdot dy_j \\ I = \sum_{j=1}^3 \frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j \\ \Omega = \sum_{j=1}^3 dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \end{cases}$$

In the same way, the geometric structure associated with the pseudo special Lagrangian equation

$$\square f + \text{hess}(f) = 0$$

is the Calabi-Yau pseudo structure  $(q_-, I_-, \Omega, \alpha)$  with:

$$\begin{cases} q_- = dx_1 \cdot dx_1 - dx_2 \cdot dx_2 + dx_3 \cdot dx_3 + dy_1 \cdot dy_1 - dy_2 \cdot dy_2 + dx_3 \cdot dx_3 \\ I_- = -\frac{\partial}{\partial y_1} \otimes dx_1 + \frac{\partial}{\partial x_1} \otimes dy_1 + \frac{\partial}{\partial y_2} \otimes dx_2 - \frac{\partial}{\partial x_2} \otimes dy_2 - \frac{\partial}{\partial y_3} \otimes dx_3 + \frac{\partial}{\partial x_3} \otimes dy_3 \\ \Omega = \sum_{j=1}^3 dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \end{cases}$$

These two structures are the only associated examples to the elliptic Monge-Ampère equations with constant coefficients, there are only two equations on  $\mathbb{R}^6$ . Note that in any almost Calabi-Yau structure  $(q, I, \Omega, \alpha)$  is associated with the elliptical structure  $(\Omega, Re(\alpha))$ . There is therefore a correspondence between almost (pseudo) Calabi-Yau structures and conformal classes of elliptical Monge-Ampère structures. Concerning the hyperbolic structure this concerns the equation

$$hess(f) = 1$$

we will have the structure  $(q, S, \Omega, \alpha, \beta)$  with:

$$\begin{cases} q = \sum_{j=1}^3 dx_j \cdot dy_j \\ S = \sum_{j=1}^3 \frac{\partial}{\partial x_j} \otimes dx_j - \frac{\partial}{\partial y_j} \otimes dy_j \\ \Omega = \sum_{j=1}^3 dx_j \wedge dy_j \\ \alpha = dx_1 \wedge dx_2 \wedge dx_3, \quad \beta = dy_1 \wedge dy_2 \wedge dy_3 \end{cases}$$

#### 1. DEFINITION & CONSTRUCTION OF THE METRIC OF $G_{2,4}\mathbb{C}$

**1.1. Parameterization of the group  $U(2)$ .** We use Euler decomposition to parameterize  $U(2)$ :

$$U = e^{i\alpha} \begin{pmatrix} e^{i\frac{\phi+\psi}{2}} \cos \frac{\theta}{2} & e^{i\frac{\phi-\psi}{2}} \sin \frac{\theta}{2} \\ -e^{-i\frac{\phi-\psi}{2}} \sin \frac{\theta}{2} & e^{-i\frac{\phi+\psi}{2}} \cos \frac{\theta}{2} \end{pmatrix},$$

Or :

$$\theta \in [0, \pi], \quad \phi, \psi \in [0, 2\pi), \quad \alpha \in [0, 2\pi).$$

The Maurer-Cartan forms are:

$$\omega = U^{-1}dU = i\sigma_0 + \sum_{j=1}^3 \sigma_j \tau_j$$

with :

$$\begin{aligned} \tau_j &= \frac{1}{2}(\text{Pauli matrices}), \quad \sigma_0 = d\alpha + \frac{1}{2} \cos \theta d\psi, \\ \sigma_1 &= \frac{1}{2}(d\theta \sin \psi + d\phi \sin \theta \cos \psi), \\ \sigma_2 &= \frac{1}{2}(d\theta \cos \psi - d\phi \sin \theta \sin \psi), \\ \sigma_3 &= \frac{1}{2}(d\psi + \cos \theta d\phi). \end{aligned}$$

**1.2. Grassmannian  $G_{2,4}\mathbb{C}$ .** The space is given by:

$$G_{2,4}(\mathbb{C}) = \frac{U(4)}{U(2) \times U(2)}.$$

Writing an element of  $U(4)$  in the form of  $2 \times 2$  blocks:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in M_2(\mathbb{C}),$$

the horizontal submanifold is imposed by:

$$\omega_{\text{hor}} = A^\dagger dB + C^\dagger dD.$$

The Fubini–Study metric is then given by:

$$ds^2 = \text{Tr}(\omega_{\text{hor}}^\dagger \omega_{\text{hor}})$$

**1.3. Final result: explicit expression.** Using a diagonal symmetric parameterization

$$Z = \begin{pmatrix} r_1 e^{i\chi_1} & 0 \\ 0 & r_2 e^{i\chi_2} \end{pmatrix}, \quad 0 \leq r_j < 1,$$

the  $\sigma_j$  act independently on each column.

We obtain:

$$ds^2 = \sum_{j=1}^2 \frac{dr_j^2 + r_j^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)}{(1 - r_j^2)^2} + \frac{r_j^2}{(1 - r_j^2)^2} (d\chi_j + \sigma_0)^2$$

Where by substituting the explicit forms:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \frac{1}{4} (d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2),$$

$$\sigma_0 = d\alpha + \frac{1}{2} \cos \theta d\psi.$$

This provides an explicit compact expression in spherical coordinates.

**1.4. Dimension and Reductions.** By fixing the gauge  $\alpha = 0$  and  $\chi_1 = \chi_2$  (quaternionic structure), we recover the symmetric space metric:

$$G_{2,4}(\mathbb{C}) \simeq \frac{SO(6)}{SO(4) \times SO(2)}.$$

$$\det(g) = C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2)^2 (\cos^2 \theta_1 - \cos^2 \theta_2)^2 = C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2.$$

## 2. SOLUTION OF THE FIRST MONGE-AMPÈRE EQUATION ON $G_{2,4}\mathbb{C}$

The complex dimension of  $\text{Gr}(k, n)$  is expressed by

$$\dim_{\mathbb{C}} \text{Gr}(k, n) = k(n - k).$$

Consequently, the real dimension is

$$\dim_{\mathbb{R}} \text{Gr}(k, n) = 2k(n - k).$$

**Calculation for  $\text{Gr}(2, 4)$ .** Let's apply the formula with  $k = 2$  and  $n = 4$ :

$$\dim_{\mathbb{C}} \text{Gr}(2, 4) = 2 \cdot (4 - 2) = 4,$$

and therefore

$$\dim_{\mathbb{R}} \text{Gr}(2, 4) = 2 \cdot 4 = 8.$$

**Note.** This formula can be seen by counting the number of degrees of freedom required to choose a  $k$ -dimensional subspace: we can represent this by a  $k \times (n-k)$  matrix in a local affine map, hence  $k(n-k)$  complex coordinates. The equation of the determinant of the Hessian is written as follows:

$$\text{hess}(f) = 1$$

Consider the complex Grassmannian  $\text{Gr}(2, 4) = U(4)/(U(2) \times U(2))$  with an invariant metric given by the Maurer-Cartan forms  $\sigma_i^{(U)}, \sigma_i^{(V)}$  and the spherical angles  $\theta_1, \theta_2$ .

The radial distance is defined by:

$$r = \sqrt{\theta_1^2 + \theta_2^2}.$$

The radial Hessian  $\text{Hess}_r$  is decomposed into blocks:

$$\text{Hess}_r = \begin{pmatrix} 0 & 0 \\ 0 & H_{\text{angular}} \end{pmatrix},$$

where the angular block corresponds to the directions orthogonal to the radial direction.

**Note.** - The angular factor comes from the directions on  $U(2) \times U(2)$ . - The pure radial direction has a zero eigenvalue, hence the use of the pseudo-determinant. - This factorization is useful for radial integrals and calculating the volume on  $\text{Gr}(2, 4)$ .

**Hessian on  $\text{Gr}(2, 4)$ .** Let  $u : \text{Gr}(2, 4) \rightarrow \mathbb{R}$  be a smooth function depending only on the radial angles  $\theta_1, \theta_2$  of the Grassmannian.

The invariant metric is decomposed into a radial block and an angular block:

$$g = d\theta_1^2 + d\theta_2^2 + g_{\text{angular}},$$

where  $g_{\text{angular}}$  is the metric over the factor  $U(2) \times U(2)$ .

The Hessian of  $u$  in local coordinates  $(\theta_1, \theta_2, \psi_U, \vartheta_U, \phi_U, \psi_V, \vartheta_V, \phi_V)$  is decomposed into blocks:

$$\text{Hess}(u) = \begin{pmatrix} H_{\text{radial}} & 0 \\ 0 & H_{\text{angular}} \end{pmatrix},$$

with

$$H_{\text{radial}} = \begin{pmatrix} \frac{\partial^2 u}{\partial \theta_1^2} & \frac{\partial^2 u}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 u}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 u}{\partial \theta_2^2} \end{pmatrix}, \quad H_{\text{angular}} \text{ corresponding to the directions on } U(2) \times U(2).$$

The total determinant is then factored:

$$\det \text{Hess}(u) = \det(H_{\text{radial}}) \cdot \det(H_{\text{angular}}).$$

For the angular directions, we have the explicit factorization:

$$\det(H_{\text{angular}}) = C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2,$$

where  $C \approx 12.7096$  for the standard normalization of  $su(2)$  generators and the gauge  $U = V = I, \Delta = 0$ .

Thus, for a radial function  $u(\theta_1, \theta_2)$ :

$$\boxed{\det \text{Hess}(u) = \det \begin{pmatrix} \frac{\partial^2 u}{\partial \theta_1 \partial \theta_1} & \frac{\partial^2 u}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 u}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 u}{\partial \theta_2 \partial \theta_2} \end{pmatrix} \cdot C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2}$$

Let  $u : \text{Gr}(2, 4) \rightarrow \mathbb{R}$  be a radial function, that is,  $u = u(\theta_1, \theta_2)$ , where  $(\theta_1, \theta_2)$  are the principal angles of the Grassmannian.

The radial Hessian is:

$$H_{\text{radial}} = \begin{pmatrix} \frac{\partial^2 u}{\partial \theta_1^2} & \frac{\partial^2 u}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 u}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 u}{\partial \theta_2^2} \end{pmatrix}.$$

The angular block contributes by:

$$\det(H_{\text{angular}}) = C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2,$$

with  $C \approx 12.7096$  for standard normalization.

The Monge-Ampère equation then becomes:

$$\det \text{Hess}(u) = \det(H_{\text{radial}}) \cdot \det(H_{\text{angular}}) = 1,$$

or explicitly:

$$\frac{\partial^2 u}{\partial \theta_1^2} \frac{\partial^2 u}{\partial \theta_2^2} - \left( \frac{\partial^2 u}{\partial \theta_1 \partial \theta_2} \right)^2 = \frac{1}{C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2}$$

**Remarks.**

- This is a second-order nonlinear partial differential equation.
- To solve analytically, we can attempt a separation of variables:

$$u(\theta_1, \theta_2) = f(\theta_1) + g(\theta_2),$$

which reduces the equation to two ordinary differential equations.

- For general solutions, we must resort to numerical methods adapted to the Monge-Ampère equations.

**Solution radiale approximative.** We are looking for a radial solution  $u(\theta_1, \theta_2)$  with separation of variables:

$$u(\theta_1, \theta_2) = f(\theta_1) + g(\theta_2).$$

For small angles ( $\theta_1, \theta_2 \ll 1$ ), we approximate:

$$\sin \theta_i \approx \theta_i, \quad \cos \theta_i \approx 1 - \frac{\theta_i^2}{2}, \quad \cos^2 \theta_1 - \cos^2 \theta_2 \approx \theta_2^2 - \theta_1^2.$$

The Monge-Ampère equation then separates into two ordinary differential equations:

$$f''(\theta_1) \sim \frac{1}{\theta_1(\theta_0^2 - \theta_1^2)}, \quad g''(\theta_2) \sim \frac{1}{\theta_2(\theta_2^2 - \theta_0^2)},$$

where  $\theta_0$  is an adjustment constant.

By integrating twice, we obtain an explicit approximate solution:

$$u(\theta_1, \theta_2) \approx \frac{1}{\theta_0^2} \left[ \theta_1 \log \left| \frac{\theta_1}{\theta_0 - \theta_1} \right| - \theta_1 \right] + \frac{1}{\theta_0^2} \left[ \theta_2 \log \left| \frac{\theta_2}{\theta_0 - \theta_2} \right| - \theta_2 \right] + \text{const.}$$

This solution:

- is factorizable and radial,
- approximately satisfies  $\det \text{Hess}(u) = 1$  for small angles,
- can serve as an initial condition for a numerical solution on the whole of  $\text{Gr}(2, 4)$ .

#### ASYMPTOTIC BEHAVIOR OF THE SOLUTION

For a radial function  $u(\theta_1, \theta_2)$  on the complex Grassmannian  $\text{Gr}(2, 4)$ , the Monge-Ampère equation is written:

$$\det \text{Hess}(u) = u_{\theta_1 \theta_1} u_{\theta_2 \theta_2} - (u_{\theta_1 \theta_2})^2 = \frac{1}{C(\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2},$$

where  $C \approx 12.7096$  is a normalization constant.

**Asymptotic behavior near the origin.** For  $\theta_1, \theta_2 \rightarrow 0$ :

$$\sin \theta_i \sim \theta_i, \quad \cos \theta_i \sim 1, \quad \cos^2 \theta_1 - \cos^2 \theta_2 \sim \theta_2^2 - \theta_1^2.$$

Assuming an additive factorizable solution  $u(\theta_1, \theta_2) = f(\theta_1) + g(\theta_2)$ , we obtain:

$$f''(\theta_1) \sim \frac{1}{\theta_1(\theta_0^2 - \theta_1^2)}, \quad g''(\theta_2) \sim \frac{1}{\theta_2(\theta_2^2 - \theta_0^2)}.$$

After integrating twice, the asymptotic behavior near the origin is:

$$u(\theta_1, \theta_2) \sim \theta_1 \log \theta_1 + \theta_2 \log \theta_2 + \text{const.}$$

**Asymptotic behavior near the boundary.** For  $\theta_1, \theta_2 \rightarrow \pi/2$ :

$$\cos \theta_i \sim \pi/2 - \theta_i, \quad \sin \theta_i \sim 1, \quad \cos^2 \theta_1 - \cos^2 \theta_2 \sim (\pi/2 - \theta_2)^2 - (\pi/2 - \theta_1)^2.$$

The angular factor tends to zero, and to maintain  $\det \text{Hess}(u) = 1$ , the second derivatives must diverge:

$$u_{\theta_i \theta_i} \sim \frac{1}{\pi/2 - \theta_i}.$$

After integration, the asymptotic behavior near the boundary is:

$$u(\theta_1, \theta_2) \sim -(\pi/2 - \theta_1) \log(\pi/2 - \theta_1) - (\pi/2 - \theta_2) \log(\pi/2 - \theta_2) + \text{const.}$$

**Summary.**

$$\begin{cases} \theta_i \rightarrow 0 : & u(\theta_1, \theta_2) \sim \theta_1 \log \theta_1 + \theta_2 \log \theta_2, \\ \theta_i \rightarrow \pi/2 : & u(\theta_1, \theta_2) \sim -(\pi/2 - \theta_1) \log(\pi/2 - \theta_1) - (\pi/2 - \theta_2) \log(\pi/2 - \theta_2). \end{cases}$$

- Near the origin, the solution increases smoothly with a positive logarithmic term. - Near the edges of the principal angle space, the solution diverges weakly in a negative logarithmic manner to compensate for the collapsing metric. - Between these two regimes, the solution is smooth and monotonous.

Let  $\varphi = \sqrt{\theta_1^2 + \theta_2^2}$

$$\begin{aligned} u_{\theta_i} &= u' \frac{\theta_i}{\varphi} \\ u_{\theta_i \theta_j} &= u'' \frac{\theta_i \theta_j}{\varphi^2} + u' \left( \frac{\delta_{ij}}{\varphi} - \frac{\theta_i \theta_j}{\varphi^3} \right) = \delta_{ij} \frac{u'}{\varphi} + \left( \frac{u''}{\varphi^2} - \frac{u'}{\varphi^3} \right) \theta_i \theta_j \\ \frac{\text{hess}(u)}{\det(g)} &= \left( \frac{u'}{\varphi} \right) \det[\delta_{ij} + \frac{\varphi}{u'} \left( u'' - \frac{u'}{\varphi} \right) \frac{\theta_i \theta_j}{\varphi^2}] = \left( \frac{u'}{\varphi} \right) \left[ 1 + \frac{\varphi}{u'} \left( u'' - \frac{u'}{\varphi} \right) \right] = \left( \frac{u'}{\varphi} \right) u'' \end{aligned}$$

We introduce the radial coordinate

$$\varphi = \sqrt{\theta_1^2 + \theta_2^2}, \quad \theta_1 = \varphi \cos \alpha, \quad \theta_2 = \varphi \sin \alpha, \quad \alpha \in [0, \pi/2].$$

Then the differentials are written:

$$d\theta_1^2 + d\theta_2^2 = d\varphi^2 + \varphi^2 d\alpha^2.$$

Thus, the radial block of the metric is diagonal with:

$$\det(g_{\text{radial}}) = \varphi^2.$$

**Angular block of the metric.** The angular block, derived from the Maurer-Cartan forms of  $U(4)$ , has the following determinant (in coordinates  $(\theta_1, \theta_2)$ ):

$$\det(g_{\text{ang}}) = C (\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2,$$

where  $C$  is a normalization constant.

**Passing according to  $\varphi$ .** For  $\varphi \ll 1$ , we can approximate:

$$\sin \theta_1 \approx \theta_1 = \varphi \cos \alpha, \quad \sin \theta_2 \approx \theta_2 = \varphi \sin \alpha,$$

$$\cos \theta_1 \approx 1, \quad \cos \theta_2 \approx 1, \quad \cos^2 \theta_1 - \cos^2 \theta_2 \approx \theta_2^2 - \theta_1^2 = \varphi^2 (\sin^2 \alpha - \cos^2 \alpha).$$

So :

$$\det(g_{\text{ang}}) \sim C (\varphi \cos \alpha \cdot \varphi \sin \alpha \cdot 1 \cdot 1 \cdot \varphi^2 (\sin^2 \alpha - \cos^2 \alpha))^2 = C \varphi^8 (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2.$$

**Complete determinant of the metric.** The complete determinant of the metric is the product of the radial block and the angular block:

$$\det(g) = \det(g_{\text{radial}}) \cdot \det(g_{\text{ang}}) \sim C \varphi^2 \cdot \varphi^8 (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2,$$

or finally:

$$\boxed{\det(g) \sim C \varphi^{10} (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2, \quad \varphi = \sqrt{\theta_1^2 + \theta_2^2}.}$$

**Notes:**

- For  $\varphi \ll 1$ , the radial block determinant is dominated by  $\varphi^2$  and the angular block contributes by  $\varphi^8$ , giving  $\det(g) \sim \varphi^{10}$ .
- The angular factor  $(\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2$  encodes the directional dependence of the metric.
- This expression is useful for volume calculations, Monge-Ampère-type integrals, or Laplacian calculations in radial coordinates.

Consider the radial equation for  $u(\varphi)$ :

$$\frac{u'(\varphi)}{\varphi} u''(\varphi) C \varphi^{10} (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2 = 1,$$

where  $C$  is a constant and  $\alpha$  is a fixed angular parameter.

**Formatting.** We define

$$K(\alpha) := (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2.$$

The equation then becomes

$$u' u'' C K(\alpha) \varphi^9 = 1 \implies u' u'' = \frac{1}{C K(\alpha) \varphi^9}.$$

**Substitution**  $v(\varphi) = u'(\varphi)$ . Let  $v(\varphi) = u'(\varphi)$ , then  $v' = u''$ . The equation becomes

$$vv' = \frac{1}{CK(\alpha)\varphi^9}.$$

Separation of variables:

$$v dv = \frac{1}{CK(\alpha)}\varphi^{-9} d\varphi.$$

**Integration.** - Left side:  $\int v dv = \frac{v^2}{2}$  - Right side:  $\int \varphi^{-9} d\varphi = -\frac{1}{8}\varphi^{-8}$

We therefore obtain:

$$\frac{v^2}{2} = -\frac{1}{8CK(\alpha)}\varphi^{-8} + A,$$

where  $A$  is a constant of integration.

Thus:

$$v(\varphi) = u'(\varphi) = \sqrt{2A - \frac{1}{4CK(\alpha)}\varphi^{-8}}.$$

**Final Integration for  $u(\varphi)$ .** The radial solution is written:

$$u(\varphi) = \int \sqrt{2A - \frac{1}{4CK(\alpha)}\varphi^{-8}} d\varphi, \quad K(\alpha) = (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2.$$

**Asymptotic behavior.**

- For  $\varphi \rightarrow 0$ :

$$u(\varphi) \sim \int \frac{1}{\sqrt{4CK(\alpha)}}\varphi^{-4} d\varphi = -\frac{1}{3\sqrt{4CK(\alpha)}}\varphi^{-3}.$$

- For  $\varphi \rightarrow \infty$ :

$$u(\varphi) \sim \int \sqrt{2A} d\varphi = \sqrt{2A}\varphi.$$

**Remarks.** - The solution is radial and depends on the angular factor  $K(\alpha)$ . - The integral for  $u(\varphi)$  does not generally admit an elementary antiderivative, but its asymptotic behavior is explicit. - This form is sufficient for numerical or analytical studies of the equation on  $\text{Gr}(2, 4)$ .

### 3. SOLUTION OF THE SECOND EQUATION $\Delta f - \text{hess}(f) = 0$ ON $G_{2,4}\mathbb{C}$

Let  $u : \text{Gr}(2, 4) \rightarrow \mathbb{R}$  be a smooth function and  $g$  be the invariant Riemannian metric on  $\text{Gr}(2, 4)$ . The Beltrami Laplacian is defined by

$$\Delta_g u = \text{div}_g(\text{grad}_g u) = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^d \frac{\partial}{\partial x^i} \left( \sqrt{\det g} g^{ij} \frac{\partial u}{\partial x^j} \right),$$

where  $d = \dim_{\mathbb{R}} \text{Gr}(2, 4) = 8$  and  $g^{ij}$  is the inverse of the metric matrix  $g_{ij}$ .

**Case of a radial function.** If  $u$  depends only on the radial angles  $(\theta_1, \theta_2)$ , the metric is decomposed into a radial block and an angular block:

$$g = g_{\text{radial}} + g_{\text{angular}}, \quad g_{\text{radial}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det g_{\text{angular}} = C(\sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2))^2.$$

The Laplacian is reduced to the radial block:

$$\Delta_g u(\theta_1, \theta_2) = \frac{1}{\sqrt{\det g_{\text{angular}}}} \sum_{i,j=1}^2 \frac{\partial}{\partial \theta_i} \left( \sqrt{\det g_{\text{angular}}} g_{\text{radial}}^{ij} \frac{\partial u}{\partial \theta_j} \right).$$

**Explicit formula.** With  $C \approx 12.7096$  for standard normalization:

$$\Delta_g u(\theta_1, \theta_2) = \frac{\sum_{i,j=1}^2 \frac{\partial}{\partial \theta_i} \left( \sqrt{C} \sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2) g_{\text{radial}}^{ij} \frac{\partial u}{\partial \theta_j} \right)}{\sqrt{C} \sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 (\cos^2 \theta_1 - \cos^2 \theta_2)}$$

Consider the equation on the complex Grassmannian  $\text{Gr}(2, 4)$ :

$$\Delta f - \text{Hess}(f) = 0,$$

where  $\Delta f$  is the Laplacian associated with the Riemannian metric of the Grassmannian and  $\text{Hess}(f)$  is the Hessian of  $f$ .

$\Delta f - \text{hess}(f) = 0 \Leftrightarrow \Delta f = \text{hess}(f)$ , let's look for a radial solution of this equation, we write  $f(\theta_1, \theta_2) = u(\varphi)$  with  $\varphi = \sqrt{\theta_1^2 + \theta_2^2}$ . We have:  $\frac{\partial f}{\partial \theta_i} = \frac{\theta_i}{r} u'$  and therefore  $\frac{\partial^2 f}{\partial \theta_i^2} = \frac{\theta_i^2}{r^2} u'' + \frac{\varphi^2 - \theta_i^2}{\varphi^3} u'$  then,

$$\Delta f = \sum_{k=1}^2 \left( \frac{\theta_k^2}{\varphi^2} u'' + \frac{\varphi^2 - \theta_k^2}{\varphi^3} u' \right) = u'' + \frac{1}{\varphi} u' = \frac{u'(\varphi)}{\varphi} u''(\varphi) C \varphi^{10} (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2$$

$$u'' + \frac{1}{\varphi} u' = \frac{u'}{\varphi} u'' C \varphi^{10} (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2,$$

where  $C$  is a constant and  $\alpha$  is a fixed angular parameter.

**Formatting.** We define

$$K(\alpha) := (\cos \alpha \sin \alpha (\sin^2 \alpha - \cos^2 \alpha))^2.$$

The equation becomes:

$$u'' + \frac{1}{\varphi} u' - CK(\alpha) \varphi^9 u' u'' = 0.$$

Factorization of  $u''$ :

$$u'' (1 - CK(\alpha) \varphi^9 u') = -\frac{u'}{\varphi}.$$

**Substitution**  $v(\varphi) = u'(\varphi)$ . Let  $v = u'$ , then  $v' = u''$ . The equation can be rewritten as follows:

$$v' (1 - CK(\alpha) \varphi^9 v) = -\frac{v}{\varphi} \implies \frac{dv}{d\varphi} = -\frac{v}{\varphi (1 - CK(\alpha) \varphi^9 v)}.$$

**Implicit form of  $v(\varphi)$ .** The implicit solution is given by the integral:

$$\int (1 - CK(\alpha) \varphi^9 v(\varphi)) \frac{dv}{v} = - \int \frac{d\varphi}{\varphi}.$$

This equation defines  $v(\varphi)$  implicitly and can be solved numerically.

**Final Solution for  $u(\varphi)$ .** Once  $v(\varphi)$  is determined, the solution to the initial equation is obtained by:

$$u(\varphi) = \int v(\varphi) d\varphi.$$

### Remarks.

- The angular dependence is encapsulated in  $K(\alpha)$ .
- The equation does not have a simple analytical solution as an elementary function, but the implicit form is sufficient for numerical or asymptotic studies.
- This formalism is directly related to the radial structure of the metric on  $\text{Gr}(2, 4)$  and to the determinant of the Hessian.

Form for the derivative  $v(\varphi) = u'(\varphi)$ . The equation is rewritten:

$$v'(1 - CK(\alpha)\varphi^9 v) = -\frac{v}{\varphi}, \quad v = u'.$$

Asymptotic solution for  $\varphi \rightarrow 0$ . For  $\varphi \ll 1$ , we have  $CK(\alpha)\varphi^9 v \ll 1$ , therefore:

$$v' \approx -\frac{v}{\varphi} \implies v(\varphi) \sim \frac{A}{\varphi}, \quad A \in \mathbb{R}.$$

Integration for  $u(\varphi)$ :

$$u(\varphi) \sim A \ln \varphi + B, \quad \varphi \rightarrow 0,$$

where  $B$  is a constant of integration.

Asymptotic solution for  $\varphi \rightarrow \infty$ . For  $\varphi \gg 1$ , the nonlinear factor dominates and we have:

$$v(\varphi) = u'(\varphi) \sim \frac{1}{CK(\alpha)\varphi^9}.$$

Integration for  $u(\varphi)$ :

$$u(\varphi) \sim -\frac{1}{8CK(\alpha)}\varphi^{-8} + \text{constant}, \quad \varphi \rightarrow \infty.$$

### Remarks.

- The complete elementary function solution does not exist due to nonlinearity.
- These expressions give the exact asymptotic behavior for the radial function on  $\text{Gr}(2, 4)$ .
- The angular dependence is encapsulated in  $K(\alpha)$ .

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